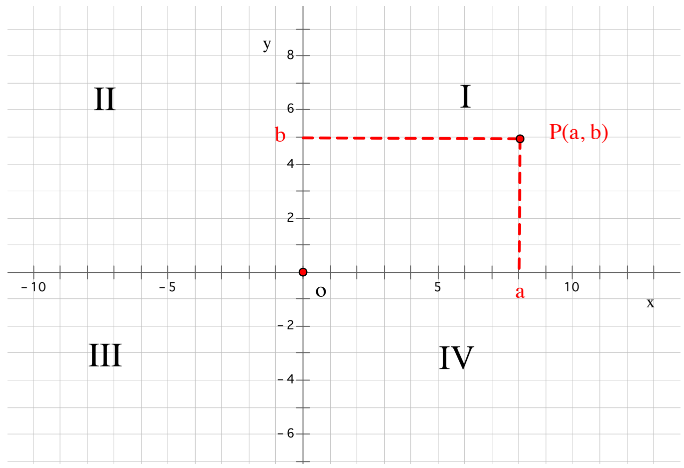
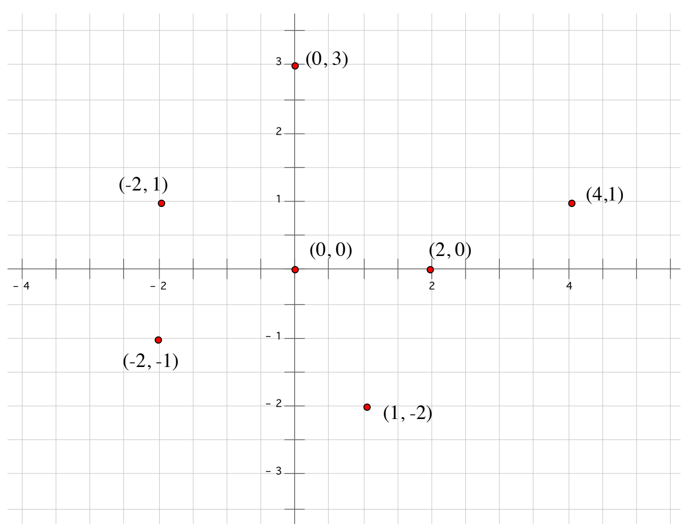
Lecture 8 : Coordinate Geometry

The coordinate plane The points on a line can be referenced if we choose an origin and a unit of distance on the axis and give each point an identity on the corresponding number line. We can also give each point in a plane an identity using an ordered pair of real numbers called Cartesian coordinates. We choose an origin o for our plane and draw a pair of perpendicular number lines intersecting at o which we call axes. Usually one of the axes is horizontal with positive numbers on the right and we call it the x-axis. The other axis is usually vertical with positive numbers above the x-axis. We usually call the vertical axis, the y-axis. This divides the plane into four **quadrants**, labelled $I - IV$ as shown below. Note that the points on the axes are not in any of the four quadrants.



Any point P in the plane can be given a unique address or label with an ordered pair of numbers $P(a, b)$. Here a is the x-coordinate which we find by constructing a line from the point to the x-axis which is perpendicular to the x-axis. The point where this line cuts the x axis is the **x-coordinate of** P (in this case a). We find the y-coordinate of the point (in this case b) by constructing a line from the point to the y-axis which is perpendicular to the y-axis. The point where this line cuts the y axis is the **y-coordinate of** P (in this case b). For a point on either axis, the coordinate corresponding to that axis is determined by their position on the axis. Here are some examples:



Conversely, given an ordered pair of numbers there is a point on the plane corresponding to that ordered pair.

Example Plot the following points on the Cartesian plane:

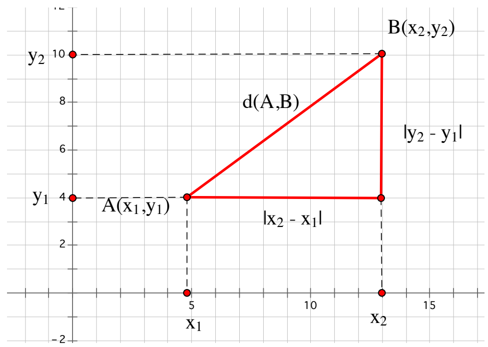
$$(0, 5) \quad (-3, 1) \quad (2, -3) \quad (-2, -2) \quad (0, -3), \quad (-3, 0).$$

Distance Formula

We can derive a formula for the distance between two point in the plane using Pythagoras' theorem. On can see form the following picture that the distance between the points A and B, denoted $d(A, B)$ is given by

$$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

where $A = (x_1, y_1)$ and $B = (x_2, y_2)$.



Example Find the distance between
(a) the points $P(-2, 1)$ and $Q(1, 1)$ and

(b) the points $P(0, 1)$ and $Q(-3, 4)$.

Graphs of Equations in Two Variables

An equation in two variables, such as $y^2 = 2x$ or $y = 3x^2 + 2x + 1$, expresses a relationship between two quantities. A point (x, y) in the Cartesian plane **satisfies the equation** if it makes the equation true when we substitute the values x and y into the equation. For example, the point $(8, 4)$ satisfies the equation $y^2 = 2x$ since $4^2 = 2(8)$. Note that the points $(1, \sqrt{2})$ and $(2, 2)$ also satisfy this equation.

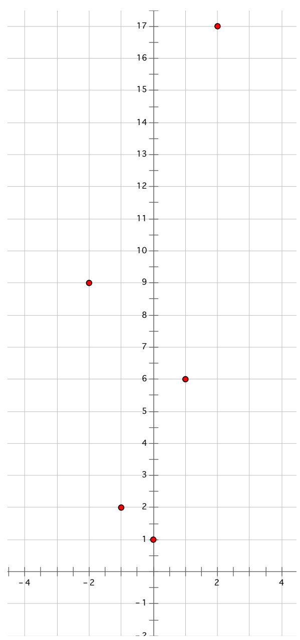
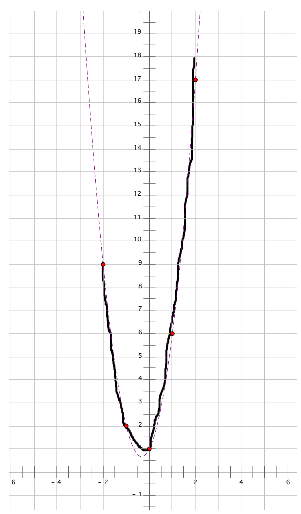
The Graph of an equation in x and y is the set of all points (x, y) in the Cartesian plane which satisfy the equation.

Below we sketch some graphs by plotting points to see general trends (and connecting the dots). Later in these lectures and in calculus, we will discuss graphing in more detail. In particular, we will learn to use the concept of limits, continuity and derivatives to get a more complete picture of what graphs look like.

Example Sketch the graph of the equation $y = 3x^2 + 2x + 1$

First we make a table which gives us some points on the graph. Since the equation gives a formula for y in terms of x, we can pick a few values of x and plug them into the equation on the right hand side to get the (unique) corresponding y value on the graph.

x	$\mathbf{y} = 3x^2 + 2x + 1$	**(x, y)**
-2	$3(-2)^2 + 2(-2) + 1 = 9$	$(-2, 9)$
-1	$3(-1)^2 + 2(-1) + 1 = 2$	$(-1, 2)$
0	$3(0)^2 + 2(0) + 1 = 1$	$(0, 1)$
1	$3(1)^2 + 2(1) + 1 = 6$	$(1, 6)$
2	$3(2)^2 + 2(2) + 1 = 17$	$(2, 17)$

On the left we show the points and on the right we've joined the dots (as best we could). The real graph is drawn as a dotted line, so we see that in this case, we get a reasonably accurate picture of this graph near the origin from our small set of data. We are left with questions that this approach to graphing cannot answer, for example:
1. "Does the graph continue to increase as we move to the right or does it turn again?"
2. "Does the graph ever cross the x-axis?"
3. "What happens to the y-values on the graph as x gets very large?".

We may be able to answer the second question later in this lecture, however we will only be able to decide the answers to questions 1 and 3 completely after studying limits and derivatives in calculus.

Example Sketch the graph of the equation $y^2 = 2x$

Example Sketch the graph of the equation $y = |x|$

Example Sketch the graph of the equation $y = x$.

Intercepts

The x-intercepts of a graph (of a given equation) are the $x-$coordinates of the points where the graph crosses the $x-$ axis. We can find the $x-$ intercepts of a graph by setting y equal to zero and solving for the values of x that satisfy the resulting equation.

The y-intercepts of a graph (of a given equation) are the $y-$coordinates of the points where the graph crosses the $y-$ axis. We can find the $y-$ intercepts of a graph by setting x equal to zero and solving for the values of y that satisfy the resulting equation.

Example Find the x and y intercepts of the graph of the following equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

and sketch its graph.

Example Find the x and y intercepts of the graph of the following equation

$$y = \frac{x^2}{9} - 1$$

and sketch its graph.

Equation of a Circle

Since a circle with center at the point (h, k) and given radius $r > 0$ is by definition equal to the set of points whose distance from the point (h, k) is r, we see that every point on such a circle satisfies the equation (using the distance formula)

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

and vice-versa. Squaring both sides of this formula, we get that every point on the circle satisfies the equation

$$(x-h)^2 + (y-k)^2 = r^2.$$

Note that the above equations are equivalent since $r > 0$ and $r = -\sqrt{(x-h)^2 + (y-k)^2}$ does not have any solutions. Thus, the equation of a circle with center (h, k) and radius $r > 0$ is given by

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

In particular, the equation of a circle centered at the origin $(0, 0)$ with radius $r > 0$ is given by

$$\boxed{x^2 + y^2 = r^2}$$

Example Sketch the graph of the equation

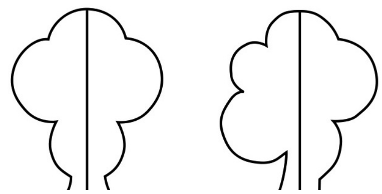
$$x^2 + 2x + y^2 + 9y + 3 = 0$$

Example Find the x and y intercepts of a circle with center $(1, 1)$ and radius 3.

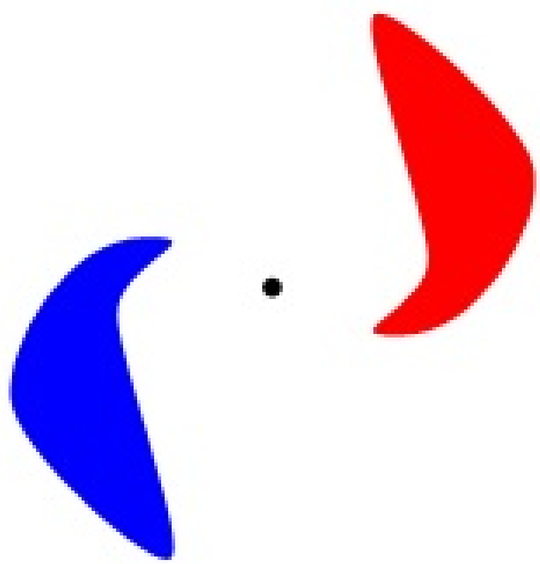
Example Where do the circles $x^2 + y^2 = 2$ and $x^2 - 2x + y^2 - 1 = 0$ intersect? Draw the graph of both circles.

Symmetry

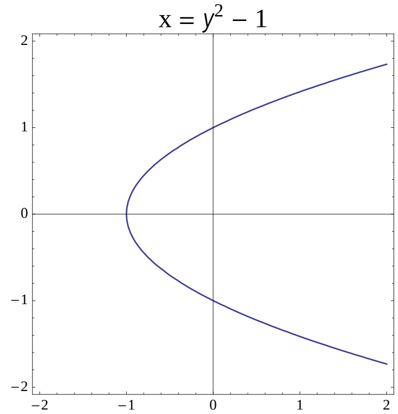
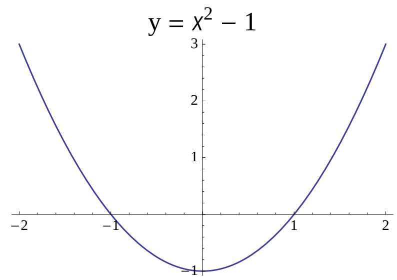
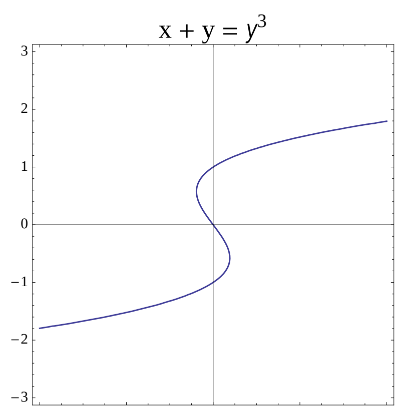
A graph can exhibit a number of types of symmetry. If one can identify which types of symmetry a graph has from the equation before graphing, it can significantly reduce the amount of work to be done in graphing. A graph has reflection symmetry in an axis if the part of the graph on one side of the axis is a reflection or mirror image of the part of the graph on the other side of the axis. For example the figure on the left below is symmetric with respect to the vertical axis shown, whereas the one on the right is not.



A graph or picture is symmetric through a point if:
when we join any point on the graph of the picture to the given point with a line and extend that line the same distance beyond the given point, the (new) point at the end of this line is also on the graph or in the picture.
This is equivalent to saying that the graph or picture remains the same when we rotate it by 180^o about the given point.
For example the picture below exhibits symmetry through the given point.



Types of Symmetry

Type of symmetry	The graph has this type of symmetry if:	What graph looks like	Geometric meaning
Symmetry with respect to $x-$axis	The equation is unchanged when y is replaced by $-y$		Graph is unchanged when reflected in $x-$axis
Symmetry with respect to $y-$axis	The equation is unchanged when x is replaced by $-x$		Graph is unchanged when reflected in $y-$axis
Symmetry with respect to the origin	The equation is unchanged when x is replaced by $-x$ and y is replaced by $-y$		Graph is unchanged when rotated by 180^o around origin

Example Test the following equations for the above symmetries and use symmetry to sketch their graphs:

$$x = |y|, \qquad y = \sqrt{9 - x^2}, \qquad 8y = x^3.$$